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On the geometry of some p-adic period spaces

Joint work with Miaofen Chen and Xu Shen

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Question: $\mathcal{G}$ p-divisible group $/\overline{\mathbb{F}}_p$, (D, φ) = rational Dieudonné module

φ: σ-linear auto.

D: finite dim. $\breve{\mathbb{Q}}_p$-v.s., $\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{un}} \circlearrowleft \sigma$

Ex.: $\mathcal{G} = A[p^\infty]$ A ab. var. $/\overline{\mathbb{F}}_p$

$D = H^1_{cris}(A/W(\overline{\mathbb{F}}_p))[\frac{1}{p}]$ $\circlearrowleft$ cristalline Frob.

$$\left\{ \begin{array}{c} p\text{-divisible groups}/\mathcal{O}_K \text{ isogenous to } \mathcal{G} \\ \text{mod. } p \end{array} \right\} \xrightarrow{\text{Hodge filtration}} \left\{ \begin{array}{c} \text{Filtrations of } D_K \\ \text{of dim. } \dim \mathcal{G} \end{array} \right\}$$

$K | \breve{\mathbb{Q}}_p$ complete-varies

p-adic period mapping.

What is the structure of the image in $\mathrm{Gr}_d(D)$, $d = \dim \mathcal{G}$,

is it an algebraic variety?

Partial answers

Drinfeld:

$$\Omega = \mathbb{P}^{n-1} \setminus \underbrace{\bigcup_{H \in \check{\mathbb{P}}^{n-1}(\mathbb{Q}_q)} H}_{\text{remove a profinite set of alg. var.}} \subset \mathbb{P}^{n-1} \quad \text{rigid analytic open subset}$$

is such a period space for p-div. gp. + particular type add. structure

Serre–Tate–Katz: Ordinary elliptic curve. $\mathcal{G} = \mu_{p^\infty} \oplus \mathbb{Q}_q/\mathbb{Z}_q$

$$\mathfrak{X} = \text{Def. space} \simeq \mathrm{Spf}\left(W(\overline{\mathbb{F}}_q)[\![x]\!]\right)$$

$$\mathfrak{X}_\eta \simeq \mathring{\mathbb{B}}^1_{\breve{\mathbb{Q}}_q} \xrightarrow{\text{period map}} \underbrace{\mathbb{A}^1}_{\text{period space for ordinary ell. curves.}} \subset \mathbb{P}^1$$

$$x \longmapsto \log(1+x)$$

Lubin–Tate–Lafaille–Gross–Hopkins: $\mathcal{G} = $ 1-dim. formal p-div. gp. height $n / \overline{\mathbb{F}}_q$

$$\mathfrak{X} = \mathrm{Def}(\mathcal{G}) \simeq \mathrm{Spf}\left(W(\overline{\mathbb{F}}_q)[\![x_1, \dots, x_{n-1}]\!]\right)$$

$$\mathfrak{X}_\eta \simeq \mathring{\mathbb{B}}^{n-1}_{\breve{\mathbb{Q}}_q} \underset{\text{étale period mapping}}{\xrightarrow{\pi}} \mathbb{P}^{n-1} = \text{period space}$$

surjective.

Fontaine: Characterize the K-points of the period space inside Gr when $[K:\mathbb{Q}_p]<+\infty$

$\rightarrow$ weak admissibility condition.

More generally: Rapoport-Zink $\mathcal{G}/\overline{\mathbb{F}}_p$ + P.E.L. additional structure

(P: polarization; E: endomorphism)

Construct: $\mathcal{M}$ / Spf$(W(\overline{\mathbb{F}}_p))$

formal scheme locally Spf$\left(W(\overline{\mathbb{F}}_p)[[X_1,\dots,X_n]]\langle Y_1,\dots,Y_m\rangle / \text{Ideal}\right)$

deformation space by quasi-isogenies of $\mathcal{G}$ (not only isomorphisms)

$\mathcal{M}_\eta$ = rigid analytic space$/\breve{\mathbb{Q}}_p$ $\qquad$ $V(\text{Ideal}) \hookrightarrow \mathring{\mathbb{B}}^n \times \mathbb{B}^m$

$+ \ \pi : \mathcal{M}_\eta \xrightarrow[\text{periodmorphism}]{\text{étale}} \mathcal{F}$] flag variety

$\underbrace{\mathcal{F}^a}_{\text{admissible locus}} := \mathrm{Im}(\pi)$ open in $\mathcal{F}$ = period space.

Fontaine's answer: $\mathcal{F}^a \subset \underbrace{\mathcal{F}^{wa}}_{\text{open}} \subset \mathcal{F}$

($\mathcal{F}^{wa}$: weakly admissible locus)

open $= \mathcal{F} \setminus$ profinite set of Schubert varieties.

s.t. $\mathcal{F}^a(K) = \mathcal{F}^{wa}(K)$ if $[K:\breve{\mathbb{Q}}_p] < +\infty$

But in general: $\mathcal{F}^a \subsetneq \mathcal{F}^{wa}$, for example $\mathcal{F}^a(\mathbb{C}_p) \neq \mathcal{F}^{wa}(\mathbb{C}_p)$

Ex: X Berkovich space $/\mathbb{Q}_p$ $\quad x \in X(\mathbb{C}_p) \setminus X(\overline{\mathbb{Q}}_p)$

$X \setminus \{x\} \underset{\neq}{\hookrightarrow} X$ but same Tate "classical points"

More generally:

local Shimura datum
$$\begin{cases} G/\mathbb{Q}_p \text{ reductive group} \\ \mu : \mathbb{G}_{m\,\overline{\mathbb{Q}}_p} \longrightarrow G_{\overline{\mathbb{Q}}_p} \text{ minuscule cocharacter up to conjugation} \\ [b] \in B(G) = \underbrace{G(\breve{\mathbb{Q}}_p)/G\text{-conjugation}} \\ \qquad \text{Kottwitz set} = \{G\text{-isocrystals}\}/\sim \\ \text{s.t. } [b] \in B(G,\mu) \end{cases}$$

crystalline Frob.

to be explained.

One can construct (Scholze's diamond + Kedlaya-Liu + the curve):

$$\underbrace{\mathcal{M}(G,\mu,b)}_{\text{local Shimura variety.}}{}_K / \breve{E}$$

rigid analytic space for $K \subset G(\mathbb{Q}_p)$ compact open

$\breve{E}$: local reflex field

+ $\pi : \mathcal{M}(G, \mu, b)_K \xrightarrow[\text{period morphism}]{\text{étale}} \underbrace{\mathcal{F}_\mu}_{\text{twisted form of } G/P_\mu}$.

$$\mathcal{F}^a_\mu := \operatorname{Im}(\pi) \underset{\text{open}}{\subset} \mathcal{F}_\mu$$

$$\mathcal{F}^a_\mu \subset \underbrace{\mathcal{F}^{wa}_\mu}_{\mathcal{F}_\mu \setminus \text{profinite set of Schubert varieties.}}$$

Hartl: has classified all possible cases when $\mathcal{F}^a_\mu = \mathcal{F}^{wa}_\mu$ for $G = GL_n$.

Th (Conjecture of Rapoport and myself):
Suppose b basic. Then
$$\mathcal{F}^a_\mu = \mathcal{F}^{wa}_\mu \iff B(G,\mu) \text{ is fully HN decomposable}$$

→ generalization of $\underbrace{\text{isoclinic}}_{\text{one Dieudonné-Manin slope}}$

→ If Shimura datum $\rightsquigarrow (G,\mu)$ then $\underbrace{B(G,\mu)}_{\text{finite}} \subset B(G)$

Classifies the Newton strata of the reduction modulo p of the Shimura variety.

$[b] \in B(G,\mu)$ basic $\longleftrightarrow$ "supersingular" closed stratum

Uniformization $\coprod_i \Gamma_i \backslash \mathcal{M}(G,\mu,b)\big|_K \xrightarrow{\sim}$ tube over supersingular locus.

$\hookrightarrow$ "The Newton polygon of non basic elements in $B(G,\mu)$ touches

Weyl chamber $\leftrightarrow G$

the Hodge polygon defined by μ outside its extremities".

<u>Ex:</u> $G = SO(n)/\mathbb{Q}_p \longleftrightarrow \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \longleftarrow\!\!\sim\!\!\sim$ Shimura var. of abelian type associated to $SO(2,n-2)$

quasi-split

$$\mu(z) = (z, 1, \dots, 1, z^{-1})$$

Then $B(G,\mu)$ is fully HN decomposable $\Rightarrow \mathcal{F}^{a} = \mathcal{F}^{wa}$.

$\rightarrow$ for $n=21$ allows us to compute the p-adic period space of K3 surfaces with supersingular reduction.

How to prove this theorem?

Use all the work of the Indian school of vector bundles (Narasimhan, Seshadri ....).

$C|\breve{\mathbb{Q}}_p \rightsquigarrow X_C$ the curve = Dedekind scheme $/\mathbb{Q}_p$

$+ \infty \in |X_C|$ $k(\infty) = C$

There is a construction

$$\{\text{Isocrystals}\} \longrightarrow \{\text{Vector bundles}/X_C\}$$

Can be enhanced to

$$G(\breve{\mathbb{Q}}_p) \longrightarrow \{G\text{-bundles}/X_C\}$$
$$b \longmapsto \mathcal{E}_b$$

Th: $B(G) \xrightarrow{\sim} H^1_{et}(X_C, G)$

$[b] \mapsto [\mathcal{E}_b]$

Moreover b basic $\Leftrightarrow \mathcal{E}_b$ semi-stable

Now for $x \in \mathcal{F}(G,\mu)(C)$ $\leadsto$ $\mathcal{E}_{b,x}$ = modification of $\mathcal{E}_b$ at ∞ given by x.

Then: $\left[\mathcal{F}(G,\mu)^a(C) = \{ x \ / \ \mathcal{E}_{b,x} \text{ is semi-stable} \} \right]$

* G quasi-split - Fix a Borel subgroup. $\mathcal{E}/X$ G-bundle

$\mathcal{E}$ semi-stable $\iff$ $\forall P$ standard parabolic $\forall \mathcal{E}_P$ reduction of $\mathcal{E}$ to P

$\forall \chi \in X^*(P)^+$ $\quad \deg(\chi_* \mathcal{E}_P) \leq 0$.

x weakly admissible $\iff$ $\forall P$ $\forall \underbrace{b_P \text{ reduction of } b \text{ to } P}_{\text{sub-isocrystal}}$

$\leadsto$ $\mathcal{E}_{b_P}$ reduction of $\mathcal{E}_b$ to P

$\leadsto$ $(\mathcal{E}_{b,x})_P$ reduction of $\mathcal{E}_{b,x}$ to P

reduction to parabolics transfer via modifications.

$\forall \chi \in X^*(P)^+$ $\quad \deg((\mathcal{E}_{b,x})_P) \leq 0$.

admissible: test on all reductions of $\mathcal{E}_{b,x}$ to P

weakly " " : " " reductions coming from a reduction of b.